CHAPTER 18. THEORETICAL FOUNDATIONS

18.1 GENERAL DESCRIPTION AND METHOD, USED

The computational processor SP LIRA 10 is designed to solve linear and nonlinear static and dynamic problems of continuum mechanics. The theory is presented in [18.3–18.5, 18.7, 18.11, 18.14, 18.29, 18.39, 18.41, 18.47, 18.48, 18.54, 18.56, 18.60, 18.62, 18.66, 18.71, 18.75, 18.79]. The Finite Element Method (FEM) is applied. The main convergence theorems and FEM error estimates are proved in [18.15, 18.17–18.20, 18.28, 18.44, 18.46, 18.49, 18.50, 18.58, 18.59, 18.80–18.88], the study of finite elements (FE) used in SP LIRA 10 is carried out in [18.33, 18.59, 18.78, 18.81, 18.82].

Structures for calculation:
- flat and spacial trusses and frames;
- plates;
- massive bodies;
- combined systems.

Loads:
- which are distributed over a region or face, specified in the global coordinate system, in which the node coordinates are defined, or in the local coordinate system associated with the element;
- the concentrated (nodal), specified in the global coordinate system or in the local coordinate system associated with the node;
- thermal, set for an element, only in a static task;
- for dynamic tasks — impulse, periodic impact, wind, seismic.

Boundary conditions, including the inhomogeneous ones, are set to determine the location and rotation of points in the global coordinate system.

Absolutely rigid bodies (ARBs) are defined by non-intersecting sets of nodes.

SP LIRA 10 implements the INSTALLATION system, which allows to mount and (or) demonstrate groups of linear and non-linear FE at each stage in accordance with the sequence of design calculation. Stability is checked after each stage.

In SP LIRA 10 there is a system **Variation of models**, which calculates the expected variations of stresses for topologically equivalent schemes with different stiffness characteristics, and the BRIDGE system, used to build lines and calculate results for moving loads.

18.2 Linear static problem

The solution U of a linear static problem for all possible displacements V satisfies the equalities of the principle of possible displacements:

$$a_0(U,V) = q(V), \quad (18.2.1)$$

where $a_0(U,V)$, $q(V)$ — are the functionals of possible work of internal and external forces, linear in *V*, the functional $a_0(U,V)$ is linear in *U* as well, symmetric and is positively defined.

FEM reduces the problem (18.2.1) to a system of linear algebraic equations (SLAE), solving which by the Gauss method, the displacements of the nodes can be determined. To speed up the solution of the SLAE, the unknowns are renumbered, which reduces the filling of the matrix [18.53]. Stresses (forces) are calculated further for each FE according to known formulas of the theory of elasticity.

18.3 Linear dynamic problem

The solution U of a linear dynamic problem for all possible displacements V satisfies the equalities:

$$b(U'',V) + c(U',V) + a_0(U,V) = q(V), \quad (18.3.1)$$

where $b(U,V)$, $c(U,V)$ — are symmetric positive definite functionals of possible work of inertial forces and forces of resistance to movement. Displacements and external forces depend on time *t*, and the dashes denote differentiation with respect to *t*. Initial or periodic conditions are added: $U(0) = U^0$, $U'(0) = U^1$, $U(0) = U(T_0)$, $U'(0) = U'(T_0)$, where T_0 is the period.

Problems (18.3.1) are solved in SP LIRA 10 by the Fourier expansion method in terms of natural vibrations [18.36, 18.38, 18.74], recommended by construction norms. Forms $V_k(x)$ and frequencies ω_k of natural oscillations are solutions of the eigenvalue problem:

$$a_0(V,W) = \omega^2 b(V,W), \quad (18.3.2)$$

and are determined by the subspace iteration method [18.53] using a modified Jacobi method. Problem (18.3.1) reduces to second-order independent ordinary differential equations with constant coefficients, which are easily solved analytically. The accuracy of the method depends on the number of computed shapes. The influence of non-calculated shapes is not taken into account in SP LIRA 10. For impulse, periodic action, wind and seismic, further calculations are made according to the design standards recommendations.

Problems (18.3.1) with initial conditions are solved in SP LIRA 10 also in the DYNAMICS+ system by the finite difference method according to an unconditionally stable scheme [18.9, 18.10, 18.25]:

$$b(\gamma_n U,V) + c(\beta_n U,V) + a_0(\alpha_n U,V) = q_n(V), \quad (18.3.3)$$

θ — time step, $t_n = n\theta$, $U_n = U(t_n)$,

$\delta_n U = (U_{n+1} - U_n)/\theta,\ \beta_n U = (U_{n+1} - U_{n-1})/2\theta = (\delta_n(U) + \delta_{n-1}(U))/2,$

$\gamma_n U = (U_{n+1} - 2U_n + U_{n-1})/\theta^2,\ \alpha_n U = (U_{n+1} + U_{n-1})/2.$

The load is a piecewise linear function of time.

The scheme error (18.3.3) is proportional to θ^2.

18.4 PROBLEMS OF ELASTIC STABILITY OF AN UNDEFORMED SCHEME

Problems of elastic stability of an undeformed scheme [18.1, 18.2, 18.27] are similar to the eigenvalue problem (18.3.2). The critical values λ_k and the corresponding forms of buckling $V_k(x)$ are solutions to the problem:

$$a_0(V,W) + \lambda a'_\sigma(U,V,W) = 0, \quad (18.4.1)$$

where $a'_\sigma(U,V,W)$ is the stability functional [18.27, 18.52], depending on the stresses (forces) obtained as a result of solving the linear static problem.

The smallest of the positive λ_k is called the stability factor or the critical value for a given load. Problems (18.4.1) are solved in SP LIRA 10 by the method of subspace iterations. The implemented methods make it possible to study not only the compressive stability but also flexural-torsional forms of buckling.

18.5 NONLINEAR STATIC PROBLEMS

SP LIRA 10 allows solving non-linear static and dynamic problems: geometrically non-linear, non-linear elastic, elastic-plastic, with one-sided constraints, including friction, problems of mechanics of granular medium (soils). The existence and uniqueness of a solution to nonlinear problems were studied in [18.8, 18.16, 18.31, 18.32, 18.35, 18.37, 18.55, 18.61, 18.68].

The solution *U* of a nonlinear static problem for all possible displacements *V* satisfies the equalities:

$$a(U,V) = q(V), \quad (18.5.1)$$

The functional *a(U,V)* of the possible work of internal forces is linear in *V* and not linear in *U*.

Non-linear static problems with continuously differentiable non-linearities (geometric non-linearity, non-linear elasticity) are solved by the step method [18.37, 18.51]:

$$a'(U_n, U_{n+1} - U_n, V) = (\theta_{n+1} - \theta_n) q(V), \quad (18.5.2)$$

где $a'(U,W,V)$ — is the derivative of $a(U,V)$,

$U_0 = 0,$

$n = 1, \ldots, N,$

$0 = \theta_0 < \theta_1 < \ldots < \theta_N = 1.$

Automatic step selection is used. The criterion is the change in geometry and stiffness. If the derivative $a'(U,W,V)$ is positively defined, the construction is stable and the error of the method (18.5.2) is proportional to the maximum step. The step method for geometrically nonlinear problems allows one to investigate the stability of a deformed circuit [18.52].

SP LIRA 10 implements the calculation of geometrically nonlinear problems after buckling: for a load at which buckling occurs, a stable equilibrium state is determined, after which the calculation continues with a step method.

When solving the geometrically nonlinear and stability problems, the presence of ARB is taken into account in accordance with [18.52].

18.6 Problems with one-sided constraints, friction and problems of a granular medium mechanics

Problems with one-sided constraints, as well as elastoplastic, with friction and problems of mechanics of a granular medium, are formulated as:

$$a_0(U,V) + d(U,V) = q(V), \tag{18.6.1}$$

где $d(U,V) = a(U,V) - a_0(U,V)$,

and are solved using the iterative method [18.12]:

$$a_0(U_{n+1},V) + d(U_n,V) = q(V). \tag{18.6.2}$$

18.7 Nonlinear dynamic problems

Since the Fourier method is not applicable to nonlinear dynamic problems, such problems are solved in SP LIRA 10 in the DYNAMICS+ system using similar (18.3.3) difference schemes [18.12, 18.25].

18.8 Finite elements of a linear static problem

In accordance with [18.59], the description of a finite element must contain:

- the task, for the solution of which it is intended;
- the domain Ω occupied by the finite element and its nodes X_l;
- a number of nodal unknowns;
- a number H_μ of linear combinations of basic functions μ_k or their explicit form.

The basic functions depend only on the geometric characteristics of the element and the order of the derivatives m in the functional of the possible work of internal forces. In the presence of bending or constrained torsion, the order of the derivatives is equal to two, in other cases, to one. Constructions are manufactured for standard FE Ω_0. The basic functions and the order of error τ are given in [18.33, 18.59, 18.73].

Let us denote $C^k(\Omega)$ as the set of k times continuously differentiable functions on Ω; $P_r(\Omega)$ as the set of polynomials of degree at most r on Ω; $Q_r(\Omega)$ as the set of products of polynomials of degree at most r in each variable, $P_r(\Omega) \subset Q_r(\Omega)$.

Rod FE

Domain FE(Ω_0) — segment [0, l] with nodes $X_1 = 0,\ X_2 = l$; nodal unknowns $u(X_i)$ at $m = 1$, $u(X_i), u'(X_i)$ at $m = 2$. The basic functions μ_k satisfy homogeneous equilibrium equations of order 2m, and therefore give an exact solution to the problem. The basic functions at a constant cross-section have the form ($s = x_1/l$):

at $m = 1$

$$\mu_1 = 1 - s,\quad \mu_2 = s, \tag{18.8.1}$$

at $m = 2$

$$\mu_1 = 1 - 3s^2 + 2s^3,\ \mu_2 = l(s - 2s^2 + s^3),\ \mu_3 = 3s^2 - 2s^3,\ \mu_4 = l(-s^2 + s^3). \tag{18.8.2}$$

For constrained torsion, the basic functions are linear combinations of polynomials of the first degree, and hyperbolic sine and cosine.

Two-dimensional FEs

Ω_0 — triangle with vertices (0, 0), (1, 0), (0, 1), unit square, quadrilateral with vertices (1, 0), (0, 1), (a, 0), (0, b), a<0, b<0. The linear transformation converts the corresponding Ω_0 into an arbitrary triangle, rectangle or convex quadrilateral. The quadrilateral is divided by diagonals into four triangles Ω_q. Knots are at the vertices, if specified, and at the midpoints of the sides.

Three-dimensional FEs

Ω_0 are single tetrahedron, cube, right triangular prism and tetrahedral pyramid. The nodes are at the vertices, if specified, and at the midpoints of the edges.

Two- and three-dimensional elements at m=1

The nodal unknowns for all elements are $u(X_i)$.

1. Triangle, $\tau = 1$:

$H_\mu = \mathrm{P}_1(\Omega)$. The basic functions on Ω_0 have the form:

$$\mu_1 = 1 - s_1 - s_2,\ \mu_2 = s_1,\ \mu_3 = s_2. \tag{18.8.3}$$

2. Rectangle, $\tau = 1$:

$H_\mu = \mathrm{Q}_1(\Omega)$. The basic functions on Ω_0 have the form:

$$\mu_1 = r_1 r_2,\ \ \mu_2 = s_1 r_2,\ \ \mu_3 = r_1 s_2,\ \ \mu_4 = s_1 s_2,\ \ r_i = 1 - s_i. \tag{18.8.4}$$

3. Quadrilateral:

Basic functions satisfy the conditions:

$$\mu_k \in P_2(\Omega_q),\ \ \mu_k \in C^1(\Omega),\ \ H_\mu \supset P_1(\Omega),\ \ \tau = 1.$$

4. Triangle with nodes in the midpoints of the sides, $\tau = 2$:

$H_\mu = \mathrm{P}_2(\Omega)$. The basic functions on Ω_0 have the form:

$$\mu_1 = 1 - 3s_1 - 3s_2 + 2s_1^2 + 4s_1 s_2 + 2s_2^2,\ \mu_2 = -s_1 + 2s_1^2,\ \mu_3 = -s_2 + 2s_2^2,$$
$$\mu_4 = 4s_1 s_2,\ \mu_5 = 4s_2 - 4s_1 s_2 - 4s_2^2,\ \mu_6 = 4s_1 - 4s_1 s_2 - 4s_1^2. \tag{18.8.5}$$

5. Quadrilateral with nodes in the midpoints of the sides, $\tau = 2$:

The basic functions satisfy the conditions: $\mu_k \in P_2(\Omega_q),\ \mu_k \in C^1(\Omega),\ H_\mu = P_2(\Omega)$.

6. Tetrahedron, $\tau = 1$:

$H_\mu = \mathrm{P}_1(\Omega)$. The basic functions on Ω_0 have the form:

$$\mu_1 = 1 - s_1 - s_2 - s_3,\ \mu_2 = s_1,\ \mu_3 = s_2,\ \mu_4 = s_3. \tag{18.8.6}$$

7. Parallelepiped, $\tau = 1$:

$H_\mu = \mathrm{P}_1(\Omega)$. The basic functions on Ω_0 have the form:

$$\mu_1 = r_1 r_2 r_3,\ \mu_2 = s_1 r_2 r_3,\ \mu_3 = r_1 s_2 r_3,\ \mu_4 = s_1 s_2 r_3,$$
$$\mu_5 = r_1 r_2 s_3,\ \mu_6 = s_1 r_2 s_3,\ \mu_7 = r_1 s_2 s_3,\ \mu_8 = s_1 s_2 s_3,\ r_i = 1 - s_i. \tag{18.8.7}$$

8. Triangular prism, $\tau = 1$:

$H_\mu \supset \mathrm{P}_1(\Omega)$. The basic functions on Ω_0 have the form:

$$\mu_1 = (1 - s_1 - s_2) r_3,\ \mu_2 = s_1 r_3,\ \mu_3 = s_2 r_3,$$
$$\mu_4 = (1 - s_1 - s_2) s_3,\ \mu_5 = s_1 s_3,\ \mu_6 = s_2 s_3, r_3 = 1 - s_3 \tag{18.8.8}$$

9. Tetrahedral pyramid, $\tau = 1$: $H_\mu = \mathrm{P}_1(\Omega) + \{x_1 * x_2/(1 - x_3)\}$.

10. Tetrahedron with nodes in the midpoints of edges, $\tau = 2$:

$H_\mu = \mathrm{P}_2(\Omega)$.

11. Parallelepiped with nodes in the middle of edges, $\tau = 2$:

$P_2(\Omega) \subset H_\mu \subset P_4(\Omega)$.

12. Triangular prism with nodes in the midpoints of the edges, $\tau = 2$:

$P_2(\Omega) \subset H_\mu \subset P_3(\Omega)$.

13. Tetrahedral pyramid with nodes in the middle of the edges, $\tau = 2$:

$$H_\mu = \mathrm{P}_2(\Omega) + \{x_1^2 * x_2/(1-x_3), x_1 * x_2^2/(1-x_3), x_1 * x_2/(1-x_3), x_1 * x_2 * x_3/(1-x_3)\}.$$

For the elements10, 11 и 12 with nodes on the edges, the equations of the faces that do not contain the X_k, have the form $l_{ki}(s_1, s_2, s_3) = 0$, where $l_{ki}(s_1, s_2, s_3)$ re polynomials of the first degree. For the vertex of such faces, one is less than for the middle of the edge. An additional polynomial for a vertex corresponds to the equation of a plane passing through the midpoints of the edges containing this vertex. Then the basic functions are the products of the constructed polynomials and the coefficient, which is calculated from the condition $\mu_k(X_k) = 1$. The equalities $\mu_k(X_i) = 0,\ i \neq k$ are satisfied by construction.

The basic functions of elements 6–9 transform the standard FE into an arbitrary one.

Two-dimensional bending elements at $m = 2$

Nodal unknowns — $u(X_i)$, $\alpha_1(X_i) = \partial u/\partial x_2(X_i)$, $\alpha_2(X_i) = -\partial u/\partial x_1(X_i)$.

14. Rectangle, $\tau = 2$:

$H_\mu \supset P_3(\Omega)$. The basic functions on the rectangle Ω_0 have the form:

$$\mu_i = \phi_i(1-s_1, 1-s_s),\ \mu_{i+3} = \phi_i(s_1, 1-s_s),\ \mu_{i+6} = \phi_i(1-s_1, s_s),\ \mu_{i+9} = \phi_i(s_1, s_s),$$
$$\phi_1 = s_1 s_2(-1 + 3s_1 + 3s_2 - 2s_1^2 - 2s_2^2),\ \phi_2 = s_1 s_2^2(1-s_2),\ \phi_3 = s_1^2 s_2(1-s_1). \quad (18.8.9)$$

15. Triangle, $\tau = 1$:

Basic functions satisfy the following conditions: $P_4(\Omega) \supset H_\mu \supset P_2(\Omega)$.

16. Quadrilateral, $\tau = 1$:

Basic functions satisfy the following conditions: $\mu_k \in P_3(\Omega_q)$, $\mu_k \in C^1(\Omega)$, $H_\mu \supset P_2(\Omega)$.

17. Triangle with nodes in the midpoints of the sides, $\tau = 1$:

Basic functions satisfy the conditions:$P_4(\Omega) \subset H_\mu \subset P_5(\Omega)$.

18. Quadrilateral with nodes in the midpoints of the sides, $\tau = 2$:

Basic functions satisfy the conditions: $\mu_k \in P_5(\Omega_q)$, $\mu_k \in C^2(\Omega)$, $H_\mu \supset P_3(\Omega)$.

The calculation of the basic functions of quadrilaterals 3, 5 and two-dimensional bending elements 14–17 is reduced to solving systems of linear equations, which is performed by the program.

18.9 Functionals of virtual works of a linear problem

Let us denote Ω as a three-dimensional domain with boundary B, $x \in \Omega$ are the independent variables, $U(x)$ are displacements, V, W are possible displacements, $\alpha(x)$ are rotations, f are external forces, $\sigma_{ij}(U)$ are stresses, $\varepsilon_{ij}(U)$, $e_{ij}(U)$ are linear and geometrically non-linear deformations, $E,\ G,\ K,\ \mu,$ are Young's moduli, shear, volumetric deformation and Poisson's ratio, $0 <= \mu < 1/2$, $G = E/(1+\mu)$, $K = E/(1-2\mu)$, $\Lambda = \mu/(1-2\mu)$, ρ is density, T is temperature, ζ is thermal deformation coefficient. The summation over repeated indices is used, the indices $i,\ j,\ k,\ l$ take the values 1, 2, 3. For a three-dimensional problem, plates and rods, the functionals of possible works are given:

- internal forces $a_0(U, V)$;
- external forces (q, V);

- temperature $g_T(V)$;
- inertial forces $b(U,V)$;
- functional for the stability problem $a'_\sigma(U,V,W)$.

Three-dimensional problem

Linear deformations are related to displacements by Cauchy dependencies:

$$\varepsilon_{ij}(U) = (\partial U_i/\partial x_j + \partial U_j/\partial x_i)/2. \quad (18.9.1)$$

Hooke's law in the orthotropic case has the form:

$$\sigma_{ij}(U) = G(\varepsilon_{ij}(U) + \delta_{ij}\Lambda\varepsilon_{kk}(U). \quad (18.9.2)$$

The functionals have the form:

$$a_0(U,V) = \int_\Omega G(\varepsilon_{ij}(U) + \delta_{ij}\Lambda\varepsilon_{kk}(U))\,\varepsilon_{ij}(V)dx, \quad (18.9.3)$$

$$(q,V) = \int_\Omega f_i U_i dx, \quad (18.9.4)$$

$$b(U,V) = \int_\Omega \rho U_i V_i dx, \quad (18.9.5)$$

$$g_T(V) = \int_\Omega K\zeta T e_{ii}(V)dx. \quad (18.9.6)$$

$$a'_\sigma(U,V,W) = \int_\Omega \sigma_{ij}(U)\cdot d^2 e_{ij}(V,W)dx, \quad (18.9.7)$$

где $d^2 e_{ij}(V,W) = (\partial V_k/\partial x_i \cdot \partial W_k/\partial x_j + \partial W_k/\partial x_i \cdot \partial V_k/\partial x_j)/2$.

Plates

The x_3 axis is orthogonal to the middle plane of the plate, which occupies the two-dimensional domain Ω_0 with the boundary B_0. Plate thickness δ, $I = \delta^3/12$, axes x_1 and x_2 lie in the median plane, $E_0 = E/(1-\mu^2)$. Indices $r,\ q$ r, q take the values 1, 2. The functional of the possible work of internal forces has the form:

$$a_0(U,U) = a_N(U,U) + a_M(U,U) + a_Q(U,U), \quad (18.9.\ 8)$$

где $a_N(U,U) = \int_{\Omega_0} \delta(E_0(\frac{\partial u_1}{\partial x_1} + \mu\partial u_2/\partial x_2)\,\partial u_1/\partial x_1 + E_0\left(\frac{\mu\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right)\partial u_2/\partial x_2 +$

$$+G(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1})^2/2)dx_1dx_2, \quad (18.9.9)$$

$$a_M(U,U) = \int_{\Omega_0} I(E_0(\partial\alpha_2/\partial x_1 + \mu\partial\alpha_1/\partial x_2)\,\partial\alpha_2/\partial x_1 +$$

$$+E_0(-\mu\partial\alpha_2/\partial x_1 + \partial\alpha_1/\partial x_2)\partial\alpha_1/\partial x_2 + G(\partial\alpha_2/\partial x_2 - \partial\alpha_1/\partial x_1)^2/2)dx_1dx_2 \quad (18.9.10)$$

$$a_Q(U,U) = k\int_{\Omega_0} \delta\, G((\partial u_3/\partial x_1 + \alpha_2)^2 + (\partial u_3/\partial x_2 - \alpha_1)^2)dx_1dx_2/2, k = 5/6. \quad (18.9.11)$$

Functional (18.9.11) takes into account the effect of transverse shear.

In the absence of transverse shear, i.e. with $a_Q(U,U) = 0$, we get:

$$a_M(U,U) = \int_{\Omega_0} I(E_0(\partial^2u_3/\partial x_1^2 + \mu\partial^2u_3/\partial x_2^2)\,\partial^2u_3/\partial x_1^2 +$$

$$+E_0(\mu\partial^2u_3/\partial x_1^2 + \partial^2u_3/\partial x_2^2)\partial^2u_3/\partial x_2^2 + 2G(\partial^2u_3/\partial x_1\partial x_2)^2)dx_1dx_2, \quad (18.9.12)$$

With the absence of bending, i.e. for $a_M(U,U) = a_Q(U,U) = 0$, we get $a_0(U,U) = a_N(U,U)$.

The functional $a_N(U,U)$ was obtained for a plane stress state, i.e. $\sigma_{33} = 0$. In the case of plane deformation, i.e. $\varepsilon_{33} = 0$, в (18.9.9) first $E_0 = E(1-\mu^2)$ is set and then μ is replaced by $\mu/(1-\mu)$.

When taking into account the elastic foundation, a term is added to $a_0(U,U)$:

$$c_0(U,U) = \int_{\Omega_0}(C_1u_3^2 + C_2(\partial u_3/\partial x_1)^2 + C_2(\partial u_3/\partial x_2)^2)dx_2dx_1, \quad (18.9.13)$$

where C_1, C_2 are the bed coefficients of the Pasternak model.

The functional $a_0(U,U)$ does not contain the rotation $\alpha_3 = (\partial u_2/\partial x_1 - \partial u_1/\partial x_2)/2$, thus in [18.90] it was proposed to add a term to $a_N(U,U)$:

$$\int_{\Omega_0} G\delta(\alpha_3 - (\partial u_2/\partial x_1 - \partial u_1/\partial x_2)/2)^2 dx_1 dx_2. \quad (18.9.14)$$

Then,

$$(g,U) = \int_{\Omega_0} (p_i u_i + m_2\alpha_2 + m_3\alpha_3) dx_1 dx_2, \quad (18.9.15)$$

$$b(U,U) = \int_{\Omega_0} \rho(\delta u_i u_i + I(\alpha_2^2 + \alpha_3^2)) dx_1 dx_2, \quad (18.9.16)$$

The temperature distribution over the thickness is linear: $T(x) = T_1 - x_3 T_3$, тогда

$$g_T(U) = \int_{\Omega_0} K_1 \zeta(\delta T_1(\partial u_1/\partial x_1 + \partial u_2/\partial x_2) + I T_3(\chi_{11} + \chi_{22})) dx_1 dx_2,\ K_1 = E/(1-\mu). \quad (18.9.17)$$

Functional for the stability problem:

$$a'_\sigma(U,V,V) = \int_\Omega [N_{rq}\partial v_3/\partial x_p \cdot \partial v_3/\partial x_q + N_{11}(\partial v_2/\partial x_1)^2 + N_{22}(\partial v_1/\partial x_2)^2 +$$
$$+N_{12}(\partial v_1/\partial x_1 \cdot \partial v_1/\partial x_2 + \partial v_2/\partial x_1 \cdot \partial v_2/\partial x_2) + 2\partial(M_{1r}\alpha_1)/\partial x_r \cdot \partial v_2/\partial x_1 -$$
$$-2\partial(M_{2r}\alpha_2)/\partial x_r \cdot \partial v_1/\partial x_2 - 2(m_2\alpha_1\alpha_3 - m_1\alpha_2\alpha_3) + m_{33}(\alpha_1^2 + \alpha_2^2) -$$
$$-\partial(M_{rq}\alpha_r\alpha_3)/\partial x_q] dx_1 dx_2 /2. \quad (18.9.18)$$

where $\alpha_1 = \partial u_3/\partial x_2$,

$\alpha_2 = -\partial u_3/\partial x_1$,

$N_{iq} = \int_{-\delta/2}^{\delta/2} \sigma_{iq} dx_3$,

$M_{rq} = -\int_{-\delta/2}^{\delta/2} x_3 \sigma_{rq} dx_3$,

$m_2 = \int_{-\delta/2}^{\delta/2} x_3 f_1 dx_3$,

$m_1 = -\int_{-\delta/2}^{\delta/2} x_3 f_2 dx_3$.

The first sum in (18.9.18) is the bending due to compression, the last ones are the influence of bending.

Terms

$N_{11}(\partial v_2/\partial x_1)^2 + N_{22}(\partial v_1/\partial x_2)^2 + N_{12}(\partial v_1/\partial x_1 \cdot \partial v_1/\partial x_2 + \partial v_2/\partial x_1 \cdot \partial v_2/\partial x_2)$ are significant though when using the method of expansion in terms of a small parameter (thickness), they are assumed to be small [18.60, 18.89]. For example, problems of stability of centrally compressed rods with sections - π, I, Z etc., can also be modeled with plates. If in (18.9.18) the specified terms are not taken into account, the critical force increases by 30–50%.

The terms containing α_3 should be introduced into (18.9.18) only if the functional $a_0(U,U)$ of the work of internal forces contains (18.9.14).

Rods

The axis x_1 is directed along the rectilinear axis of the rod, the segment $[0,l] = \Omega_0$, the axes x_2, x_3 are the main central axes of the section A, $|A|$ — sectional area, $I_2 = \int_A x_3^2\, dA$, $I_3 = \int_A x_2^2\, dA$ — inertia moments, x_2^0, x_3^0 — coordinates of the center of torsion. The primes denote differentiation with respect to x_1, I_ω is the sectorial moment of inertia.

$$a_0(U,U) = \int_0^l (E|A|u_1'^2 + EI_2 u_3''^2 + EI_3 u_2''^2 + GI_1\alpha_1'^2/2 + EI_\omega \alpha_1''^2 + 2(EI_3 u_2''')^2/GF_3) +$$
$$+2(EI_2 u_3''')^2/GF_2) dx. \quad (18.9.19)$$

The elastic foundation is modeled by a rectangle orthogonal to the x_3 axis. The width of the rectangle b corresponds to the contact surface. A term is added to $a_0(U,U)$:

$$c_0(U,U) = b\int_0^l (C_1(u_3^2 - 2x_2^0 u_3\alpha_1 + (x_2^0)^2\alpha_1^2 + b^2\alpha_1^2/12) +$$
$$+C_2(u'^2_3 - 2x_2^0 u'_3\alpha'_1 + (x_2^0)^2\alpha'^2_1 + b^2\alpha'^2_1/12 + \alpha_1^2))dx_1. \quad (18.9.20)$$

For the orthogonal x_2 axis, the bases are replaced in (18.38) u_3 by u_2 and x_2^0 by $-x_3^0$.

$$(g,U) = \int_0^l (p_i u_i + (m_1 + f_2 x_3^0 - f_3 x_2^0)\alpha_1 - m_2 u'_3 + m_3 u'_2 + m_\omega \alpha'_1)dx_1, \quad (18.9.21)$$

$$b(U,U) = \int_0^l \rho(|A|(u_i u_i + 2(x_3^0 u_2 - x_2^0 u_3)\alpha_1 + R\alpha_1^2) + I_2 u_3'^2 + I_3 u_2'^2 + I_\omega \alpha'^2_1)dx_1, \quad (18.9.22)$$

где $R = (x_2^0)^2 + (x_3^0)^2 + (I_2 + I_3)/|A|$,

$$m_1 = \int_A (x_2 f_3 - x_3 f_2)\, dA + p_2 x_3^0 - p_3 x_2^0,$$
$$m_2 = \int_A x_3 f_1\, dA,$$
$$m_3 = -\int_A x_2 f_1\, dA, \quad (18.9.23)$$
$$p_i = \int_A f_i\, dA,$$
$$m_\omega = \int_A f_1\, \varphi dA.$$

The temperature distribution over the cross-section is linear: $T(x) = T_1 - x_2 T_2 - x_3 T_3$, тогда

$$g_T(U) = \int_0^l E\zeta(T_1|A|u'_1 + T_2 I_3 u''_2 + T_3 I_2 u''_3)dx_1. \quad (18.9.24)$$

Functional for the stability problem is:

$$a'_\sigma(U,V,V) = \int_l [N_1(\alpha_2^2 + \alpha_3^2) + M_1(\alpha_2{}'\alpha_3 - \alpha_3{}'\alpha_2) - 2(M_2\alpha_1)'\alpha_3 + 2(M_3\alpha_1)'\alpha_2 +$$
$$+(M_2\alpha_1\alpha_3)' - (M_3\alpha_1\alpha_2)' - m_2\alpha_1\alpha_3 + m_3\alpha_1\alpha_2 +$$
$$+(N_1 r^2 + M_2 I_{32} - M_3 I_{23} + M_\omega I_{33})\alpha_1^2]dx_1/2 - d^2(f,V), \quad (18.9.25)$$

где $r^2 = (I_2 + I_3)/A$,

$$I_{33} = \int_A \varphi(x_2^2 + x_3^2)\, dA/I_\omega,$$
$$I_{32} = \int_A x_3(x_2^2 + x_3^2)\, dA/I_2,$$
$$I_{23} = \int_A x_2(x_2^2 + x_3^2)\, dA/I_3,$$
$$N_i = \int_A \sigma_{1i}\, dA,$$
$$M_1 = \int_A (x_2\sigma_{13} - x_3\sigma_{12})\, dA,\ M_2 = \int_A x_3\sigma_{11}\, dA,\ M_3 = -\int_A x_2\sigma_{11}\, dA.$$

The first term under the integral in (18.9.25) is compressive flexure, the second is torsion flexure, the third-eighth is flexural torsion, and the last is compressive torsion. This term should be introduced only if the functional $a_0(U,U)$ contains $EI_\omega\alpha'^2_1$ under the integral, otherwise, the critical force may be underestimated. The last three terms in (18.9.25) take into account the asymmetry of the load in the section, for example, the force is not applied at the center of gravity.

Absolutely rigid body (ARB)

All ARB deformations are equal to zero, so the functional $a_0(U,U)$ is zero. Then,

$$b(U,U) = \rho(|\Omega|u_i u_i + I_k\alpha_k^2),$$
$$(f,U)_\Omega = p_i u_i + m_i\alpha_i,$$

where $p_i = \int_\Omega f_i\, d\Omega,\ m_1 = \int_\Omega (x_2 f_3 - x_3 f_2)\, d\Omega,\ m_2 = \int_\Omega (x_3 f_1 - x_1 f_3)\, d\Omega,\ m_3 = \int_\Omega (x_1 f_2 - x_2 f_1)\, d\Omega.$

$$a'_\sigma(U,V,V) = m_{ij}\alpha_i\alpha_j/2,$$

where $m_{ij} = -\int_\Omega x_i f_j\, d\Omega,\ \ i \neq j,\ \ m_{ii} = \int_\Omega x_k f_k\, d\Omega$, summing over $k \neq i$.

If ARB is a segment, and $x_2 = x_3 = 0$, then $m_{12} = -m_3$, $m_{13} = m_2$, $m_{23} = m_{11} = 0$.

18.10 FINITE ELEMENT METHOD FOR A LINEAR STATIC PROBLEM

All the considered problems (nonlinear, dynamic, for eigenvalues) are reduced to solving a sequence of linear ones. Linear problems are defined as the principle of possible displacements:

$$a(U,V) = q(V), \quad (18.10.1)$$

where $a(U,V)$ is a symmetric positive definite bilinear functional;

$q(V)$ is a linear functional, the real displacement U and any possible displacement V are defined on the domain Ω with boundary B and belong to the energy space H.

Problem (18.10.1) must be reduced to a finite-dimensional one, to a system of linear algebraic equations (SLAE). One of the most versatile and common ways to do this is the finite element method (FEM) in displacements.

The domain Ω is divided into finite elements (FE) Ω_r, which, depending on the dimension, are segments, convex polygons or polyhedra, $B_r$is the boundary of Ω_r. Different FEs do not have common interior points. The nodes X_j of the finite element mesh are the vertices FE, nodes on the sides (edges) are also possible. Let us denote the maximum distance between grid nodes belonging to the same FE as h.

The division into finite elements is assumed to be consistent: if a vertex or edge of an element also belongs to another FE, then it is a vertex or edge of this other finite element.

All functionals obtained by integration over Ω will be represented as the sums of the corresponding integrals over Ω_r, the functional obtained by integration over Ω_r, is denoted by index r.

The unknown FEMs (degrees of freedom) are the linear functionals $L_k(U)$, whose carriers we denote by S_k. The functionals $L_k(U)$ are linearly independent: if the equalities $c_k L_k(U) = 0$ are satisfied for all $U \in H$, then all $c_k = 0$. Basically, the functionals $L_k(U)$ are the values of the functions and their derivatives at the nodes, then S_k coincides with one of the nodes.

Unknowns in nodes:

- displacements for 3D domains;
- movements and rotations for bars and plates;
- for the thin-walled bars, a seventh displacement is added.

The joining of all the elements Ω_r, containing S_k is called the star of elements Ω^k, which corresponds to the functional $L_k(U)$.

The displacements are approximated by linear combinations of the basic functions $\mu_k(x)$:

$$U_h(x) = d_k \mu_k(x). \quad (18.10.2)$$

The set of functions of the form (15.10.2) is denoted by H_μ.

The basic functions $\mu_k(x)$ are nonzero only on Ω^k, and they satisfy the equalities:

$$L_k(\mu_i) = \delta_{ki}. \quad (18.10.3)$$

From (18.10.3) follows the linear independence of the basic functions.

From (18.10.3) with $U = U_h$, $V = \mu_i$, the FEM equations are obtained:

$$a(U_h, \mu_i) = q(\mu_i), \quad (18.10.4)$$

which, using (18.10.2), can be written in the form:

$$d_k a(\mu_k, \mu_i) = q(\mu_i). \quad (18.10.5)$$

The elements of the matrix $a(\mu_k, \mu_i)$ and the vector $q(\mu_i)$ are obtained by summing over all FE Ω_r the elements of the matrices $a_r(\mu_k, \mu_i)$ and the vectors $q_r(\mu_i)$. Obviously, $a_r(\mu_k, \mu_i) \neq 0$ only for $\Omega_r \subset \Omega^k \cap \Omega^i$, $q_r(\mu_k) \neq 0$ only for $\Omega_r \subset \Omega^k$.

The matrix $a_r(\mu_k, \mu_i)$ and the vector $q_r(\mu_i)$ are calculated in the local coordinate system associated with the element, corrected for hinges by Jordan eliminations, for rigid insertions - by means of the matrix corresponding to ARB displacements, and then are converted to the general one; a matrix composed of the coordinates of the unit vectors of the local system and the matrix of cosines are used.

When calculating integrals, numerical integration is applied, namely, the cubature formulas:

$$\int_\Omega f d\,\Omega = \gamma_k f(x_k),\ x_k \in \Omega. \tag{18.10.6}$$

The described method for constructing the system of FEM equations is rather simple and is based on the fact that most of the calculations are performed independently on each FE, which is one of the main algorithmic advantages of FEM.

The second significant advantage of FEM is the ease of satisfying the boundary conditions. From (18.10.2) and (18.10.3) the equality $L_j(U_h) = d_j$. Therefore, the homogeneous boundary condition $L_j(U_h) = z_j$ is not necessarily required, it is sufficient to set $d_j = z_j$ in (18.10.2) and in (18.10.5). The boundary conditions for stresses (forces) are always met since FEM uses the principle of possible displacements.

The elements of the matrix $a(\mu_k, \mu_i)$ are nonzero only if the intersection $\Omega^k \cap \Omega^i$ contains at least one FE. Such matrices are called sparse or weakly filled. When solving the system (18.10.5) by the Gauss method, the fill-in (the number of non-zero elements) increases. To reduce the number of calculations and the time of solving the SLAE, the unknowns should be renumbered so that the fill-in becomes as small as possible. Such methods, based on graph theory are described in [18.53].

By solving the problem (18.10.5), the displacements of each FE in its coordinate system can be found, then the stresses for three-dimensional elements are calculated, and the forces for rods and plates. The forces in the bars are corrected for distributed loads.

18.11 CALCULATION OF STIFFNESS CHARACTERISTICS OF THE ROD SECTION

Algorithms for calculating the cross-sectional area, the coordinates of the center of gravity, the position of the main central axes and moments of inertia are well known. To calculate the characteristics for torsion and shear, it is required [18.41] to solve the equations in section A:

$$-(\partial(G\psi_i/\partial x_2)/\partial x_2 - (\partial(G\psi_i/\partial x_3)/\partial x_3 + p_i = 0 \qquad (18.11.1)$$

with boundary conditions

$$G\partial\psi_i/\partial x_2 n_2 + G\partial\psi_i/\partial x_3 n_3 + q_i = 0, \qquad (18.11.2)$$

где i = 1,2,3, $p_1 = 0$, $p_2 = Ex_2, p_3 = Ex_3, q_1 = -Gx_3n_2 + Gx_2n_3$,

$q_2 = G\mu[(x_2^2 - x_3^2)n_2 + 2x_2x_3n_3]/4, \quad q_3 = G\mu[2x_2x_3n_2 + (x_3^2 - x_2^2)n_3]/4$,

where E, G, μ are Young's modulus, shear modulus and Poisson's ratio, which can be variable over the section area.

The Neumann problem (18.11.1), (18.11.2) has a unique (up to an additive constant) solution if $\int_\Gamma qn_j\, d\Gamma = 0$. These equalities follow from Green's formula:

$$\int_\Gamma qn_j\, d\Gamma = \int_A \partial q/\partial x_j\, dA.$$

To apply FEM, we define the problem (18.11.1), (18.11.2) in a form, similar to the principle of possible displacements:

$$\int_A (G(\partial\psi_i/\partial x_2 * \partial v_i/\partial x_2 + \partial\psi_i/\partial x_3 * \partial v_i/\partial x_3) + p_i v)\, dA + \int_\Gamma q_i v d\,\Gamma = 0. \qquad (18.11.3)$$

Suppose that $\varphi = \psi_1 - x_3^0 x_2 + x_2^0 x - c$, где x_2^0,x_3^0 — are the coordinates of the torsion center.

Having found ψ_i из (4.7), we determine x_2^0, x_3^0 and c from the conditions:

$\int_A Ex_2\varphi\, dA = \int_A Ex_3\varphi\, dA = \int_A E\varphi\, dA = 0$:

$$x_2^0 = -\int_A Ex_3\psi_1\, dA/EI_2,\quad x_3^0 = \int_A Ex_2\psi_1\, dA/EI_3,\quad \mathrm{c} = \int_A E\psi_1\, dA/E|A|. \qquad (18.11.4)$$

Then the moment of inertia of torsion EI_2 and the sectorial moment of inertia $\mathrm{E}I_\omega$ are calculated:

$$\mathrm{GI}_1 = \int_A G((\partial\psi_1/\partial x_2 - x_3)^2 + (\partial\psi_1/\partial x_3 + x_2)^2)dA,\quad \mathrm{EI}_\omega = \int_A E\varphi^2\, dA. \qquad (18.11.5)$$

Next, using the functions ψ_2,ψ_3 the shear areas can be found:

$$\mathrm{GF}_2 = (EI_2)^2/\int_A G(\eta_{22}^2 + \eta_{23}^2)dA,\quad \mathrm{F}_3 = (EI_3)^2/\int_A G(\eta_{32}^2 + \eta_{33}^2)dA. \qquad (18.11.6)$$

$$\eta_{22} = \mu(x_2^2 - x_3^2)/4 + \partial\psi_2/\partial x_2,\quad \eta_{23} = \mu x_2 x_3/2 + \partial\psi_2/\partial x_3,$$

$$\eta_{32} = \mu x_2 x_3/2 + \partial\psi_3/\partial x_2,\quad \eta_{33} = \mu(x_3^2 - x_2^2)/4 + \partial\psi_3/\partial x_3.$$

The method, described applies to thin-walled sections as well. The final elements are rectangles and the desired function on each rectangle is represented as:

$$\psi(y_2, y_3) = \psi_1(y_2) + y_3\psi_2(y_2), \qquad (18.11.7)$$

where the axes y_2, y_3 are directed along the long and short, "thin-walled" sides. Substituting (18.11.7) into (18.11.3), a variational formulation for a thin-walled section can be got.

This approach is universal and does not require different algorithms for open, closed, semi-closed, etc. sections.

The formulas for calculating stresses at an arbitrary point in the section are:

$$\sigma_{11} = Eu_1' - x_2u_2'' - x_3u_3'' + \alpha_1''\varphi,$$

$$\sigma_{12} = G(\alpha_1'(\partial\psi_1/\partial x_2 - x_3)/2 + u_2'''\eta_{22} + u_3'''\eta_{32}),$$

$$\sigma_{13} = G(\alpha_1'(\partial\psi_1/\partial x_3 + x_2)/2 + u_2'''\eta_{23} + u_3'''\eta_{33}).$$

18.12 STATIONARY HEAT CONDUCTIVITY PROBLEM

The heat equation [18.16] is derived from the law of conservation of energy and the Fourier law, in the stationary case it has the form:

$$-div(KgradT) = q, \quad (18.12.1)$$

where T is temperature;

q is the density of heat sources;

K — is the thermal conductivity coefficient;

$Q = -KgradT$ the heat flow (Fourier law).

The application of FEM requires an integral identity similar to the principle of possible displacements. It is obtained from (18.12.1) and Green's formula for integration by parts:

$$\int_\Omega q * t * d\Omega = \int_\Omega -div(KgradT) * t * d\,\Omega = \int_\Omega KgradT * gradt * d\Omega - \int_\Gamma K\partial T/\partial n * t * d\Gamma \quad (18.12.2)$$

Here t is a variation of T.

Three are three variants of possible boundary conditions:

1) The temperature $T = T_0$ is set on the boundary section;

2) On part of the boundary the heat flux $Q = Q_0$ is set

3) On part of the boundary B_3, heat exchange with the environment takes place according to the law:

$$Q + k(T - T_1) = 0,$$

where k is the heat transfer coefficient, T_1 — is the ambient temperature;

are implemented similarly to the given displacements

Boundary condition 2) is implemented similarly to the load on a surface or a line.

Boundary condition 3) is implemented similarly to elastic support with stiffness k, in addition, a load $k \cdot T_1$.

18.13 STATIONARY FILTRATION PROBLEM

The filtration equation [18.16] is obtained from the law of conservation of mass and Darcy's law, in the stationary case it has the form:

$$-div(Kgradh) = 0, \quad (18.13.1)$$

$$h = p/\rho g + z, \quad (18.13.2)$$

where p is pressure;

h is height of liquid;

ρ is the density of the liquid;

K is the filtration coefficient;

$v = -Kgradh$ filtration rate (Darcy's law).

The application of FEM requires an integral identity similar to the principle of possible displacements. It is obtained from (18.13.1) and Green's formula for integration by parts:

$$0 = \int_\Omega -div(Kgradh) * q * d\,\Omega = \int_\Omega Kgradh * gradq * d\Omega - \int_\Gamma K\partial h/\partial n * q * d\Gamma. \quad (18.13.3)$$

Here q is a variation of p.

From the equality to zero of the integral over B, we obtain the boundary conditions:

- on an impenetrable surface $K\partial h/\partial n = 0$, this condition does not need to be set in FEM as it will be fulfilled automatically;

- on the free surface:

$$p = 0. \tag{18.13.4}$$

It is more convenient to set the boundary conditions for pressure rather than for height of liquid. The inhomogeneous condition (18.13.4) is also easily realized.

Substituting (18.13.2) into (18.13.3), we obtain the FEM equation:

$$\int_\Omega K/(\rho g) gradp * gradq * d\Omega + \int_\Omega K\partial q/\partial z * d\Omega = 0. \tag{18.13.5}$$

The second term in (18.13.5) is "load". From (18.13.5) and the boundary conditions we find the pressure.

The resulting pressure is transmitted for further calculation of the soil mass, the dependence for stresses is used:

$$\sigma_{ij} = \sigma_{ij}^0 + p\delta_{ij}.$$

That is, the pressure will give an additional load, which is calculated similarly to the temperature one. When calculating, stresses: $\sigma_{ij}^0 = \sigma_{ij} - p\delta_{ij}$ are analyzed.

18.14 NON-STATIONARY PROBLEM FOR THERMAL CONDUCTIVITY

In matrix form, the non-stationary heat conduction equation is written as:

$$[\mathrm{C}] \cdot \frac{\partial}{\partial t}\{T\} + [K] \cdot \{T\} = \{F\} \tag{18.14.1}$$

where $[K]$ – a positively defined symmetric matrix of thermal conductivity coefficients, or simply, a thermal conductivity matrix;

$[C]$ is the heat capacity matrix;

$\{T\}$ and $\{F\}$ are the temperature and right-hand side vectors, respectively.

SP LIRA 10 uses an implicit integration scheme:

$$[\mathrm{C}] \cdot \frac{T_{i+1} - T_i}{\Delta\tau} + [K] \cdot T_{i+1} = F_i \tag{18.14.2}$$

where $\Delta\tau$ is the time step (discretization step);

T_i, T_{i+1} are the vectors of temperatures at the current and next moments of time;

F_i is the vector of the right side at the current time.

Boundary conditions for the problem of non-stationary heat conduction are similar to the boundary conditions implemented for stationary heat conduction (see Section 18.12), but with the ability to assign their variability in the time domain.